Representations of Cherednik Algebras

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The product rule yields $\frac{\partial}{\partial x}x = x\frac{\partial}{\partial x} + 1$ and $\frac{\partial}{\partial x_2}x_1 = x_1\frac{\partial}{\partial x_2}$. Consider the algebra generated by $x_1, x_2, \ldots, x_n$ (multiplication by variables) and $\partial_1, \partial_2, \ldots, \partial_n$ (differentiation), subject to $[x_i, \partial_j] = x_i\partial_j - \partial_j x_i = -\delta_{ij}$, i.e. -1 if $i = j$ and 0 otherwise.

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It is apparently useful in certain fields, like the representation theory of S_n.

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D_1 x_1^2 =& \frac{\partial}{\partial x_1} x_1^2 - c \sum_{i \neq 1} (x_1 - x_i)^{-1} (1 - S_{1i}) x_1^2 \\
=& 2x_1 - c \sum_{i \neq 1} (x_1 - x_i)^{-1} (x_1^2 - x_i^2) \\
=& 2x_1 - c \sum_{i \neq 1} (x_1 + x_i) \\
=& 2x_1 - c(n-1)x_1 - c \sum x_i
\end{aligned}
$$

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For any V, there is an automatic representation of $\mathrm{End}\ V$ with the identity map, and for any A, with A an algebra, there is a obvious representation of A with $\rho(x)(y) = 0$ for all $x \in A$, $y \in V$. Finally, if your algebra is a field, a representation is just a vector space

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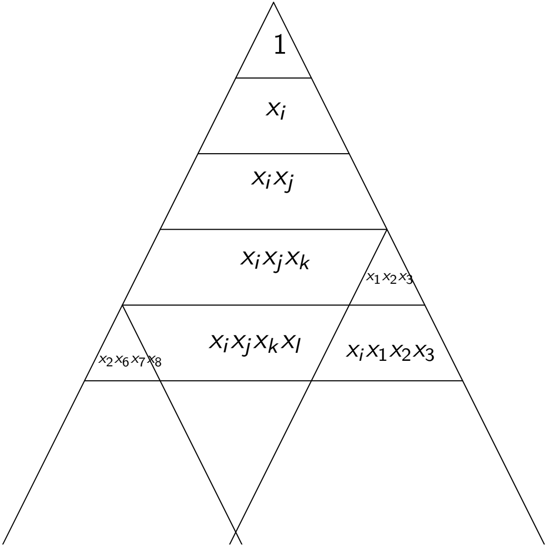
In quantum mechanics, we can adjust the $\frac{\partial}{\partial x_i}$ term in Dunkl operators by a factor of $\hbar$. Then, the x_i are position vectors, D_i are momenta, and the extra part is accounting for Heisenberg's Uncertainty Principle.



1
x_i
$x_i x_j$
$x_i x_j x_k$
$x_1 x_2 x_3$
$x_2 x_6 x_7 x_8$
$x_i x_j x_k x_l$
$x_i x_1 x_2 x_3$

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Devadas and Sun: "The Polynomial Representation of the Type A_{n1} Rational Cherednik Algebra in Characteristic $p|n$" proves similar kinds of results for characteristic $p|n$ by extending certain results from Pavel to positive characteristic.

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3. We also conjecture that, for $t\geq 3$ (and showed this for $t=0,1,2$):

$$Dy_1{}^t x_1^s = \sum_{r=0}^{t}(-c)^r \frac{s!}{(s-r)!}f(r)$$

where $f(0)=\sum_{a=s-t}x_1^a$, $f(1)=\sum_{a+b=s-t}\sum_{i\neq 1}x_1^a x_i^b$, $f(2)=\sum_{a+b+c=s-t}\sum_{1\neq i\neq j\neq 1}x_1^a x_i^b x_j^c$, etc.

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My mentor, Gus Lonergan

My parents

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